Connection Matrices in *Macaulay2*

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Feynman Integrals

Feynman integrals are fundamental building blocks in the computation of **scattering amplitudes**. They satisfy a **holonomic** systems of differential equations (restricted **GKZ**), often expressed as a **Pfaffian system**.

$$I(m_i, p_{ij}; \nu) = \int_{\{\alpha_i \geq 0\}} \frac{(\alpha_1)^{\nu_1} \ldots (\alpha_m)^{\nu_m}}{(\mathcal{G}(\alpha; m_i, p_{ij}))^{D/2}} \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_m}{\alpha_m}$$

$$\partial_{\mathbf{x}} \bullet \begin{pmatrix} MI_1 \\ \vdots \\ MI_r \end{pmatrix} = \begin{pmatrix} a_{11}^{(\mathbf{x})} & \ldots & a_{1r}^{(\mathbf{x})} \\ \vdots & \ddots & \vdots \\ a_{r1}^{(\mathbf{x})} & \ldots & a_{rr}^{(\mathbf{x})} \end{pmatrix} \begin{pmatrix} MI_1 \\ \vdots \\ MI_r \end{pmatrix}, \mathbf{x} \in \{m_i, p_{ij}\}$$

Figure 1: Above: Feynman integral in Lee–Pomeransky representation. The variables refer to Schwinger parameters α_i, masses m_i, and Mandelstam variables p_{ij}. Below: Pfaffian system in terms of the master integrals MI_i

Diff.Equations to D-Modules

The Weyl algebra of **linear** differential operators with polynomial respectively rational coefficients is non-commutative:

$$D_n \coloneqq \mathbb{Q}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

$$R_n \coloneqq \mathbb{Q}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$$

A system of linear differential equations is encoded by a **left ideal** in the respective Weyl algebra:

$$P_i \bullet f = 0, \quad \forall i = 1, \ldots, k$$

$$I \coloneqq D_n \langle P_1, \ldots, P_k \rangle$$

$$R_n I \coloneqq R_n \langle P_1, \ldots, P_k \rangle$$

The solutions to the differential equations can be recovered algebraically from the associated **D**-module:

$$\mathrm{Sol} = \mathrm{Sol}(D_n/I) = \mathcal{H}om_{D_n}(D_n/I, \mathcal{O})$$

Connection Matrices

For a D-ideal with finite **holonomic rank** r, the corresponding system of differential equations can be turned into a first order **matrix differential equation** of rank r by the choice of a basis.

$$\mathrm{rank}(I) \coloneqq \dim_{\mathbb{Q}(x)} R_n / R_n I$$

$$[s_1], [s_2], \ldots, [s_r] \quad \mathbb{Q}(\mathbf{x})\text{-basis}$$

$$\partial_i \bullet \vec{F} = A_i(\mathbf{x}) \cdot \vec{F}$$

$$\vec{F} = (s_1 \bullet f, \ldots, s_r \bullet f)^\top$$

The matrices $A_1, \ldots, A_n$ are matrices with rational function entries and are called the **connection matrices** of I. They encode the action of differentiation modulo the extended ideal.

In our implemented `ConnectionMatrices` package for *Macaulay2*, the connection matrices are computed by polynomial reduction using **Gröbner bases** in the rational Weyl algebra.

Example

Pfaffian System

$$\partial_x \bullet \begin{pmatrix} 1 \\ \partial_y \end{pmatrix} = \begin{pmatrix} 0 & y/x \\ -xy & 1/x \end{pmatrix} \begin{pmatrix} 1 \\ \partial_y \end{pmatrix} \quad \text{mod } \mathbf{R}_2 I$$

$$\partial_y \bullet \begin{pmatrix} 1 \\ \partial_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -x^2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \partial_y \end{pmatrix} \quad \text{mod } \mathbf{R}_2 I$$

Gröbner Basis

$$g_1 = \underline{\partial_x^2} + y^2 \qquad g_3 = y\underline{\partial_x \partial_y} - \partial_x + xy^2$$

$$g_2 = \underline{\partial_y^2} + x^2 \qquad g_4 = x\underline{\partial_x} - y\partial_y$$

Figure 2: A Pfaffian system for $\sin(xy)$. It is generated from the ideal $\langle \partial_x^2 + y^2, \partial_y^2 + x^2 \rangle$. A Gröbner basis of $R_2 I$ is provided for the `lex` order. From the underlined leading terms one can deduce the standard monomials 1 and ∂_y.

Gröbner Bases in Weyl Algebras

Gröbner bases also exist in the **non-commutative** setting of Weyl algebras. In the context of **elimination orders**, Gröbner bases in D_n descend to Gröbner bases in R_n.

$\mathbf{D}_n$		$\mathbf{R}_n$
$\left\{ x_1^{\alpha_1} \ldots x_n^{\alpha_n} \partial_1^{\beta_1} \ldots \partial_n^{\beta_n} \right\}$		$\left\{ \partial_1^{\beta_1} \ldots \partial_n^{\beta_n} \right\}$
$\mathbb{Q}$-coefficients		$\mathbb{Q}(\mathbf{x})$-coefficients
GB for elim. order $\prec$	$\Rightarrow$	GB for $\prec\|_{\partial^\beta}$

Once a Gröbner basis of $R_n I$ is computed, the set of **standard monomials** can be conveniently read off. They provide a $\mathbb{Q}(\mathbf{x})$-basis of the rational Weyl algebra modulo the extended ideal.

The reduction of an arbitrary element in R_n to the standard monomials is carried out via the **normal form** algorithm. It allows to compare equivalence classes modulo $R_n I$ and to check ideal containment.

Computation via Normal Form

$$\texttt{normalForm}_{I,\prec} (\partial_\mathbf{i} s_\mathbf{1}) = \mathbf{a_{11}}(x) s_\mathbf{1} + \ldots + \mathbf{a_{1r}}(x) s_\mathbf{r}$$

$$\texttt{normalForm}_{I,\prec} (\partial_\mathbf{i} s_\mathbf{2}) = \mathbf{a_{21}}(x) s_\mathbf{1} + \ldots + \mathbf{a_{2r}}(x) s_\mathbf{r}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\texttt{normalForm}_{I,\prec} (\partial_\mathbf{i} s_\mathbf{r}) = \mathbf{a_{r1}}(x) s_\mathbf{1} + \ldots + \mathbf{a_{rr}}(x) s_\mathbf{r}$$

$$\Rightarrow \partial_\mathbf{i} \bullet \begin{pmatrix} s_\mathbf{1} \\ \vdots \\ s_\mathbf{r} \end{pmatrix} = \begin{pmatrix} a_\mathbf{11} & \ldots & a_\mathbf{1r} \\ \vdots & \ddots & \vdots \\ a_\mathbf{r1} & \ldots & a_\mathbf{rr} \end{pmatrix} \begin{pmatrix} s_\mathbf{1} \\ \vdots \\ s_\mathbf{r} \end{pmatrix} \quad \text{mod } \mathbf{R}_n I$$

Figure 3: The entries of the connection matrices A_i can be read off directly from the normal form of $\partial_i s_j$. Here $s_1, \ldots, s_r$ are standard monomials of $R_n I$ some chosen monomial order $\prec$.

Gauge Transformation

Connection matrices in **other bases** than standard monomials can be obtained by a **gauge transformation**.

$$\vec{F}' = g\vec{F} \quad g \in \mathrm{GL}_r(\mathbb{Q}(x))$$

$$A_i' = g A_i g^{-1} + (\partial_i \bullet g)\, g^{-1}$$

In ε-parametric examples, gauge transformations sometimes allow to obtain **ε-factorized** connection matrices.

Implementation in M2

The implementation of our `ConnectionMatrices` package allows for parametric computations over $\mathbb{Q}(\underline{\varepsilon})$ as well as custom elimination orders. The core methods are `normalForm`, `standardMonomials`, `connectionMatrices`, and `gaugeTransform`.

References

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