Connection Matrices in *Macaulay2*

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Studying PDEs algebraically

A system of **linear differential equations** with **polynomial coefficients** in one unknown function $f(x_1, \dots, x_n)$ can be written as

$$P_1 \bullet f = \dots = P_r \bullet f = 0 \tag{1}$$

where '$\bullet$' denotes the application of the operator and each P_i can be expressed using multi-indices as

$$P_i = \sum_{(\alpha,\beta)} c_{\alpha,\beta} x^\alpha \partial^\beta \quad \text{where} \quad c_{\alpha,\beta} \in \mathbb{C}. \tag{2}$$

As in the case of commutative algebra, we can then, instead of looking at the finite list of operators $P_1, \dots, P_r$ equivalently look at the **left-ideal** generated by them in the **non-commutative algebra** of all differential operators with polynomial coefficients.

The Weyl algebra

The (n-th) **Weyl algebra** of linear differential operators with polynomials coefficients over $\mathbb{C}$ is denoted by

$$D_n \coloneqq \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle.$$

It consists of the elements of the free algebra generated by $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$ modulo the relation $[\partial_i, x_i] = 1$ and the commutativity of all other generators.
Similarly, the **rational Weyl algebra** consists of the linear differential operators with rational function coefficients and is denoted by

$$R_n \coloneqq \mathbb{C}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle.$$

The system of PDEs in Equation (1) can now succinctly be written in terms of a single ***D*-ideal** as

$$I \bullet f = 0 \tag{1'}$$

where $I = D_n\langle P_1, \dots, P_r \rangle$.

This is the starting point of **algebraic analysis** which studies the properties of solutions to Equation (1) in terms of either the D-ideal I or more generally the ***D*-module** D_n/I. If instead we consider the R-module R_n/R_nI we study the properties of solutions at the 'generic point'.

Elimination term orders

A total order $\prec$ on the monomials $\{x^\alpha \partial^\beta\}_{(\alpha,\beta)\in\mathbb{N}^{2n}}$ is called an **elimination term order** if 1 is the smallest monomial and:

1. $1 \prec x_i \partial_i$,
2. $x^\alpha \partial^\beta \prec x^a \partial^b$ implies $x^{\alpha+s}\partial^{\beta+t} \prec x^{a+s}\partial^{b+t}$,
3. $\partial^\beta \prec \partial^\gamma$ implies $x^\alpha \partial^\beta \prec \partial^\gamma$.

A classical **example** of an **elimination order** is given by considering the **weight order** w.r.t. $v \in \mathbb{N}^n_{>0}$ on the ∂'s and then refining with the lex order on

$$\partial_1 \succ \dots \succ \partial_n \succ x_1 \succ \dots \succ x_n.$$

The resulting order is denoted by $\prec_{(0,v)}$.

Compatibility with localization: A Gröbner basis of a D_n ideal I w.r.t. $\prec_{(0,v)}$ descends to a Gröbner basis of R_nI w.r.t. $\prec'_{(0,v)}$, i.e. the restriction of the order to the ∂'s.

Connection matrices

When the **holonomic rank** of the D-ideal, that is

$$\operatorname{rank}(I) \coloneqq \dim_{\mathbb{C}(x)} R_n/R_nI \tag{3}$$

is finite, the differential Equation (1) can be expressed in **matrix form** using a basis $s_1, \dots, s_m$ of R_n/R_nI as

$$\partial_i \bullet \vec{F} = A_i \cdot \vec{F}, \quad i = 1, \dots, n \tag{4}$$

where $\vec{F} = (s_1 \bullet f, \dots, s_m \bullet f)^\top$.

The matrices $A_1, \dots, A_n$ are the **connection matrices** of I and encode the action of multiplying by the ∂_i's modulo R_nI. The $s_1, \dots, s_m$ can be chosen as the **standard monomials** of R_nI w.r.t. $\prec'_{(0,v)}$.

The connection matrices can be computed via the **normal form algorithm** in R_n. Let G be a **Gröbner basis** of R_nI w.r.t. $\prec'_{(0,v)}$. Then,

$$\texttt{normalForm}(\partial_k s_i, G) = \sum (A_k)_{i,j} s_j.$$

Gauge transformation

Connection matrices in a different basis, for example in terms of

$$\widetilde{F} = gF, \quad g \in \mathrm{GL}_m(\mathbb{C}(x)),$$

are obtained via **gauge transformation**:

$$\widetilde{A}_i = g A_i g^{-1} + (\partial_i \bullet g)\, g^{-1}, \quad i = 1, \dots, n. \tag{5}$$

In ε-parametric examples, gauge transformations sometimes allow to obtain **ε-factorized** connection matrices.

Example

Consider the following **ideal of differential operators** in D_2:

$$I = \langle x\partial_x^2 - y\partial_y^2 + \partial_x - \partial_y, x\partial_x + y\partial_y + 1 \rangle \eqqcolon \langle P_1, P_2 \rangle$$

Its **solution space**, that is the functions f, such that $P_1 \bullet f = 0$ and $P_2 \bullet f = 0$ is given by:

$$\mathrm{Sol}(I) = \mathbb{C}\left\langle \frac{1}{x-y}, \quad \frac{1}{x-y}\log(x/y) \right\rangle$$

Setting $v = (2, 1)$, a **Gröbner basis** for I w.r.t. $\prec_{(0,v)}$, is given by

$$G = \left\{ y\partial_x\partial_y + \partial_x + y\partial_y^2 + \partial_y, \quad x\partial_x + y\partial_y + 1, \quad xy\partial_y^2 - y^2\partial_y^2 + x\partial_y - 3y\partial_y - 1 \right\}.$$

From $\mathrm{in}_{\prec'_{(0,v)}}(R_2I) = \langle \xi_x\xi_y, \xi_x, \xi_y^2 \rangle$ we can then read off the **standard monomials**

$$s_1 = 1, \quad s_2 = \partial_y,$$

such that the system $I \bullet f = 0$ can be expressed in **matrix form** as

$$\partial_x \bullet \begin{pmatrix} f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} -\frac{1}{x} & -\frac{y}{x} \\ -\frac{1}{x(x-y)} & -\frac{x+y}{x(x-y)} \end{pmatrix} \cdot \begin{pmatrix} f \\ \partial_y f \end{pmatrix} \quad \text{and} \quad \partial_y \bullet \begin{pmatrix} f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{(x-y)y} & \frac{3y-x}{(x-y)y} \end{pmatrix} \cdot \begin{pmatrix} f \\ \partial_y f \end{pmatrix}.$$

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- **Live demo of our package**
- **Implemented methods in M2**
- Check out the **documentation** at `mathrepo.mis.mpg.de/ConnectionMatrices/`
- **Contact**: nicolas.weiss@mis.mpg.de