Exact Volumes of Semi-Algebraic Convex Bodies

MAX-PLANCK-INSTITUT FÜR MATHEMATIK IN DEN NATURWISSENSCHAFTEN

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arXiv:2602.04707

Overview

We compute volumes of semi-algebraic convex bodies defined by finitely many concave polynomials with arbitrary precision. This is motivated by geometric statistics, where intersections of convex bodies arise as maximum likelihood estimator (MLE) sets [1].

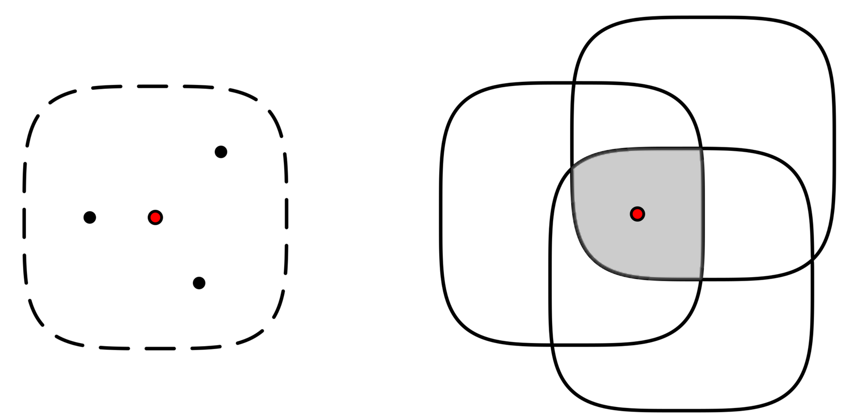

Figure: On the left: An ℓ_4-ball with unknown center θ in $\mathbb{R}^2$ and $p_1, \ldots, p_k$ samples. On the right, K is translated to each of the p_i. The MLE set is the shaded region on the right.

Consider a convex body $K + \theta$ with unknown center θ and random samples $X_1, \ldots, X_k$ from $K + \theta$. The MLE set is then the bounded convex set

$$C = \cap_{i=1}^{k}(K + X_i).$$

One can think of the MLE set as an analog of the confidence interval in higher dimensions, i.e.

$$\mathrm{vol}(C) = \mathrm{uncertainty}(\mu \mid X_1, \ldots, X_k).$$

We present an algorithm for the computation of the volume of these MLE sets based on the approach of [2]. The setting of concave polynomial, however, allows for various simplifications. We provide a full implementation of the algorithm in SageMath available on Github and Zenodo [3].

Volumes as Periods of Rational Functions

Classically, the volume of semi-algebraic sets C is computed by *counting points* in C, for example with a Monte Carlo approach. This amounts to integrate the *volume form* over C. When $C = \{f(x) > 0\}$ for f a smooth polynomial, one can also represent the volume as a period of a rational function:

$$\int_C 1 dx_1 \wedge \ldots \wedge dx_n \quad = \quad \mathrm{Vol}(C) \quad = \quad \frac{1}{2\pi i}\int_\Gamma \frac{x_1 \partial_1 \bullet f(x)}{f(x)} dx_1 \wedge \ldots \wedge dx_n.$$

Whereas in the classical view C is an *open* and *real* integration contour in $\mathbb{R}^n$, the expression as a period of a rational function is an integral over a *closed* and *complex* contour Γ in $\mathbb{C}^n$. This contour is the *Leray coboundary* for $\{x \in \mathbb{R}^n \mid f(x) = 0\}$.

For $f(t, x)$, such that $I(t) = \int_{\Gamma(t)} A(t, x)dx$, D-module theory naturally provides non-trivial differential equations for such parametric periods: Any element of the *integration ideal*

$$D_t \cap (\mathrm{Ann}_{D_{t,\mathbf{x}}}(A(t, x)) + \partial_{x_1} D_{t,\mathbf{x}} + \ldots + \partial_{x_n} D_{t,\mathbf{x}}), \quad \text{where} \quad D_{t,\mathbf{x}} = \mathbb{Q}[t, \mathbf{x}]\langle \partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\rangle,$$

annihilates $I(t)$, provided that the contour $\Gamma(t)$ is locally constant in a suitable sense. Integration ideals formalize *integration-by-parts* identities and are computed by a process called *creative telescoping*.

The research of NW is funded by the European Union (ERC, UNIVERSE PLUS, 101118787). Views and opinions expressed are, however, those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Volumes as Solutions of Differential Equations

Volumes of a convex semi-algebraic set $C = \bigcap_i \{f_i > 0\}$ can be expressed as solutions to *univariate linear differential* equations in the following two ways. It relies on realizing C as one of the regions of $\{\prod f_i > 0\}$, therefore placing it in a family of smooth varieties defined by $\prod f_i - t$.

(A) - Deformation:

$$\mathrm{Vol}(C) = \lim_{0 \leftarrow t} \mathrm{Vol}(C_t)$$

(B) - Integration:

$$\mathrm{Vol}(C_\epsilon) = \int_{a(\epsilon)}^{b(\epsilon)} \mathrm{Vol}(C_\epsilon \cap \{x_1 = s\}) ds$$

The volume of the semi-algebraic set (or region thereof) is then realized as the limit respectively the integral of a holonomic function. And the linear differential operator P that annihilates it only depends on the defining polynomials $f_1, \ldots, f_k$. If P has order$(P) = r$, then that holonomic function can be solved to *arbitrary precision* by *recursively* providing either r volumes of *deformations* respectively *lower-dimensional* slices.

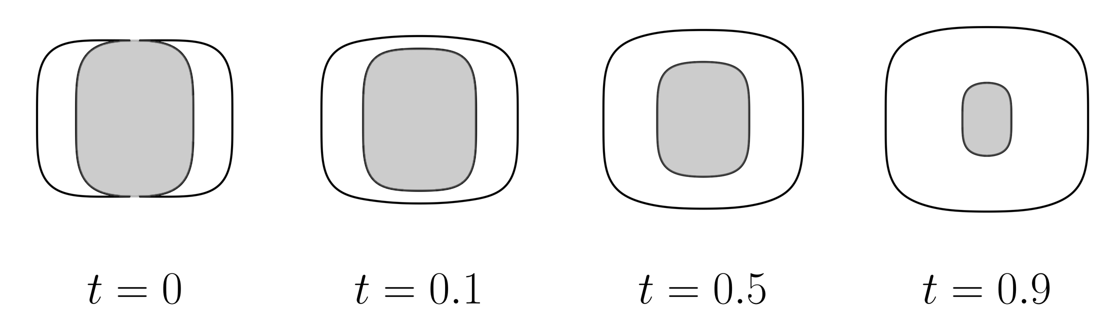

Figure: The intersection of two ℓ_4-balls in $\mathbb{R}^2$ defined by $f_1 = 1 - x^4 - y^4$ and $f_2 = 1 - (x - 1/2)^4 - y^4$ is deformed into the family of smoothly bounded sets $C_t \subset \{\prod f_i - t > 0\}$. $Vol(C_t)$ is annihilated for t small by a linear differential operator P_t of order $\mathrm{ord}(P_t) = 6$ and degree $\deg(P_t) = 18$.

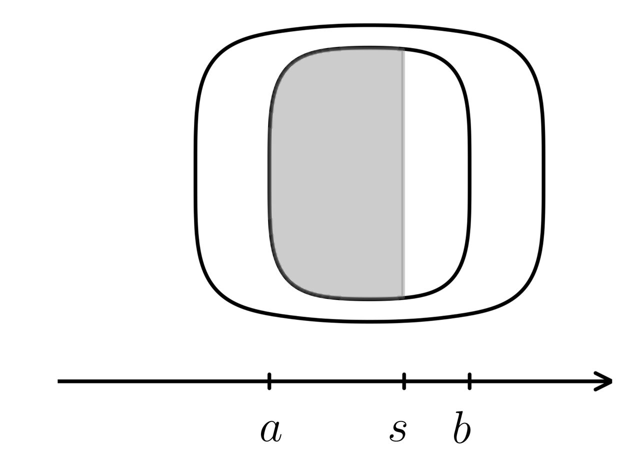

Figure: For f_1, f_2 as on the left and for generic small t_0, the volume of the one-dimensional slices $\mathrm{Vol}(C_{t_0} \cap \{x = s\})$ is annihilated by a linear differential operator P_s of order $\mathrm{ord}(P_s) = 2$ and degree $\deg(P_s) = 23$.

In the above example, the volume of the intersection is then determined by computing a total of 7 Picard–Fuchs operators and 12 one-dimensional slice volumes. Using high-precision solvers for linear ODEs one obtains to arbitrary precision Vol(C) = 2.708344826299720090001844936980643720528690610575117510163356615319 . . .

References

[1] V. Koltchinskii, L. Ramesh, and M. Wahl. *Maximum likelihood estimation of the location of a symmetric convex body.* In prepration, 2026.

[2] P. Lairez, M. Mezzarobba, and M. Safey El Din. *Computing the Volume of Compact Semi-Algebraic Sets.* In *Proceedings of ISSAC '19*. ACM, 2019.

[3] L. Ramesh and N. Weiss. *nicolas-alexander-weiss/*`exact_convex_volumes`*: v0.0.1.* https://doi.org/10.5281/zenodo.18472470

[4] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner deformations of hypergeometric differential equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.