Exact Volumes of Semi-Algebraic Convex Bodies

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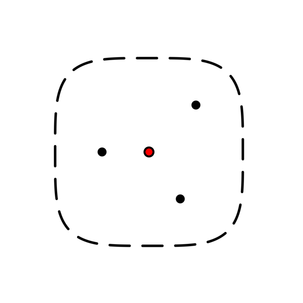
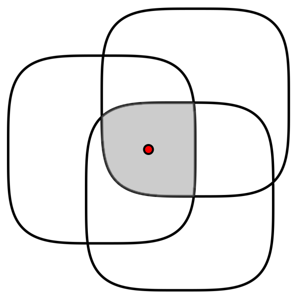
A Problem from Geometric Statistics [1]

Let $K = \{\mathbf{x} \in \mathbb{R}^n \mid 1 - x_1^p - \ldots - x_n^p > 0\}$, $p \in 2\mathbb{N}$. Given samples $X_1, \ldots, X_k$ from a uniform distribution on $K + \theta$ with $\theta \in \mathbb{R}^n$, determine θ:





$\Rightarrow$ The **M**aximum **L**ikelihood **E**stimator set $= \bigcap_i K + X_i$, is guaranteed to contain θ.

Problem: Quantify **uncertainty**$(\theta \mid X_1, \ldots, X_k)$ by determining **Vol** $(\bigcap_i K + X_i)$ with arbitrary precision.

[1] Koltchinskii, Ramesh, and Wahl. 2026. *Maximum likelihood estimation of the location of a symmetric convex body.* Manuscript in preparation.

Intersections of ℓ_p Balls are Special

Definition
We call $f \in \mathbb{Q}[x_1, \ldots, x_n]$ *concave* if it defines a concave function on $\mathbb{R}^n$.

Fact: Superlevelsets of concave functions $\{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) > c\}$ are convex.

Definition
Let $f_1, \ldots, f_k \in \mathbb{Q}[x_1, \ldots, x_n]$ be *concave* polynomials, such that $C = \bigcap_i \{\mathbf{x} \in \mathbb{R}^n \mid f_i(\mathbf{x}) > 0\}$ is full-dimensional.

We refer to such regions as ***semi-algebraic convex bodies.***

Problem: Determine the volume of *semi-algebraic convex bodies* to arbitrary precision.

Computing Volumes of Compact Semi-Algebraic Sets

Contributed Paper

ISSAC '19, July 15–18, 2019, Beijing, China

Computing the Volume of Compact Semi-Algebraic Sets

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We will recall their approach.

Semi-algebraic convex bodies allow to simplify their algorithm.

[2] Lairez, Mezzarobba, and Safey El Din. 2019. Computing the Volume of Compact Semi-Algebraic Sets. ISSAC'19. ACM, 259–266

Smooth Volumes are Periods

Let $f \in \mathbb{Q}[\mathbf{x}]$ and $V(f)$ smooth.

Let C be union of bounded connected components of $\{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) > 0\}$:

$$\mathrm{Vol}(C) = \int_C 1 d\mathbf{x} = \underbrace{\int_{\partial C} x_1 dx_2 \dots dx_n}_{\text{Stokes' Thm.}} = \underbrace{\frac{1}{2\pi i} \int_\Gamma \frac{(\partial_{x_1} \bullet f) x_1}{f} d\mathbf{x}}_{\text{Leray's Residue Thm.}}$$

$\Gamma \subset \mathbb{C}^n - V(f)$ closed contour (Leray coboundary).

Slogan: *Smooth volumes are periods of rational functions over closed contours.*

Slice Volumes are Periods as functions in t

Let $f \in \mathbb{Q}[t, \mathbf{x}]$ and $V(f)$ smooth.

Let C be union of fiberwise bdd. conn. components of $\{(t, \mathbf{x}) \in \mathbb{R}^{1+n} \mid f(t, \mathbf{x}) > 0\}$.

Assume the slice $C_t = C \cap \{t\} \times \mathbb{R}^n$ to be smooth.

$$\mathrm{Vol}(C_t) = \int_C 1 d\mathbf{x} = \underbrace{\int_{\partial C} x_1 dx_2 \ldots dx_n}_{\text{Stokes' Thm.}} = \underbrace{\frac{1}{2\pi i} \int_{\Gamma_t} \frac{(\partial_{x_1} \bullet f) x_1}{f} d\mathbf{x}}_{\text{Leray's Residue Thm.}}$$

Proposition (LMS'19): $\mathrm{Vol}(C_t)$ is a solution of a **Picard–Fuchs** operator $P_t \in \mathbb{Q}[t]\langle \partial_t \rangle$ for t in any interval of adjacent critical values of the projection

$$\mathrm{pr}_t : V(f) \to \mathbb{R}, \qquad (t, \mathbf{x}) \mapsto t.$$

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Slogan: *Volumes can be computed by solving [3] a linear ODE to **arbitrary precision**.*

[3] Mezzarobba. 2016. *Rigorous Multiple-Precision Evaluation of D-Finite Functions in SageMath.*

Volumes of Semi-Algebraic Sets: Integration

Let $C \subset \{x \in \mathbb{R}^n \mid f(\mathbf{x}) > 0\}$ compact. $V(f)$ smooth. Set $t = x_1$.

Algorithm (paraphrasing LMS'19)

(1) Compute P_t annihilating $\phi(t) := \mathrm{Vol}(C_t)$.

(2) Compute $\{c_1, \ldots, c_r\} := \mathrm{CritVals}$ of $\mathrm{pr}_t : V(f) \to \mathbb{R}$.

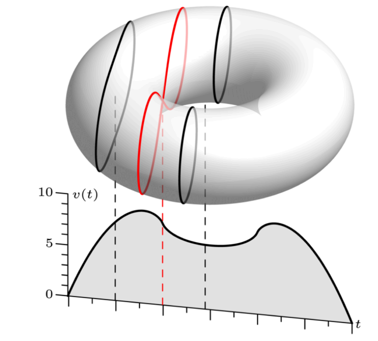
For each interval (c_1, c_2):

(3) Compute $\mathrm{Vol}(C_{t_0})$ for $\mathrm{ord}(P_t)$ many values $t_0 \in (c_1, c_2)$.

(4) Solve $P_t \partial_t \bullet \phi(t) = 0$ using the initial data (3).

$$\Rightarrow \mathrm{Vol}(C) = \sum_i \left(\int_{\mathbf{c}_1}^{t} \mathrm{Vol}(C_t) dt \right) \Big|_{\mathbf{t}=\mathbf{c}_2}$$

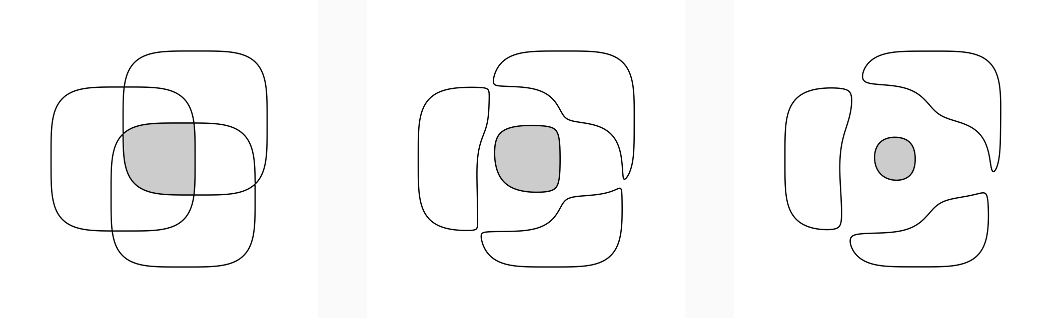
Initial Data: $\mathrm{Vol}(C_{t_0})$ *are smooth volumes of dimension* $n-1$.



[Figure from LMS'19].

Volumes of Semi-Algebraic Sets: Deformation

Deform a compact n-dimensional semi-algebraic set $C = \bigcap_i \{\mathbf{x} \in \mathbb{R}^n \mid f_i(\mathbf{x}) > 0\}$ into smooth $C_t = C \cap \{\mathbf{x} \in \mathbb{R}^n \mid \prod_i f_i(\mathbf{x}) - t > 0\}$:



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.5$

Compute P_t for $F_t := \prod_i f_i(\mathbf{x}) - t$. Solve on $0 < t << 1$ and take limit $t \to 0$.

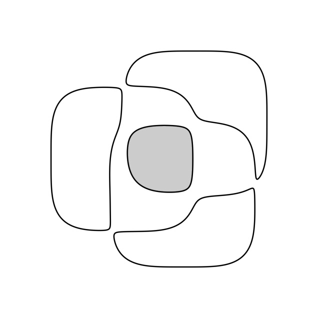
Initial Data: *Volumes* $\mathrm{Vol}(C_{t_0})$ *with smooth boundary* $F(t = t_0, \mathbf{x})$.

Observations / Problems

P1: Creative telescoping (to compute P_t) is the most expensive step in each recursion.

P2: Deformation of $C = C_1 \cap \ldots \cap C_k$ introduces new critical values (from $V(F_t)$) not corresponding to ∂C_t.

P3: Slices of deformation $C_t \cap L$ may have multiple components even though C is connected.



Solution:
Only solve P_t (and compute initial data) for those (c_1, c_2), s.t.

$$C_t \cap pr_{x_i}^{-1}(c_1, c_2) \neq \emptyset.$$

Consider only those c_i coming from ∂C_t instead of $V(f_t)$.

Deformation of Semi-Algebraic Convex Bodies

Deformations of *semi-algebraic convex bodies* are convex:

Proposition (Ramesh–W '26)
Let $f_1, \ldots, f_k \in \mathbb{Q}[x_1, \ldots, x_n]$ *be concave functions defining a semi-algebraic convex body* C*.*

Then, its deformation

$$C_t := \{x \in \mathbb{R}^n \mid \prod_i f_i(x) - t > 0\} \cap C$$

is a convex set for all $t > 0$*.*

Corollary
At each step of the recursion there is a unique interval of critical values coming from ∂C_t *on which initial data have to be computed.*

Deformations of *semi-algebraic convex bodies* are convex:

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Corollary
At each step of the recursion there is a unique interval of critical values coming from ∂C_t *on which initial data have to be computed.*

Remark 1: It is not enough to suppose that $C = C_1 \cap \dots \cap C_k$ is convex.

Remark 2: We still have to determine the relevant critical values $\subsetneq \mathrm{CritVals}(V(F_t))$.

Identifying the Relevant Critical Values from ∂C_t

Project onto x_1. $F_t := \prod_i f_i - t$. Critical locus $I := \langle F_t, \partial_2 F_t, \ldots, \partial_n F_t \rangle \subset \mathbb{Q}[\mathbf{x}]$.

Case $\dim I = 0$:

Case $\dim I > 0$:

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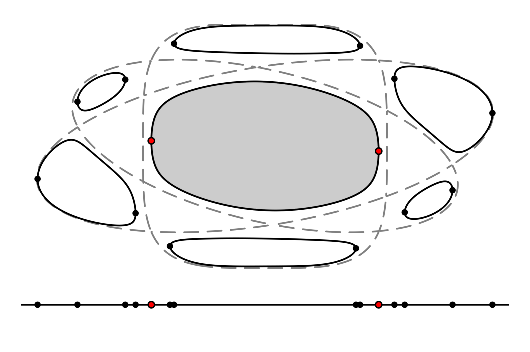
Case $\dim I = 0$:

1. $\mathrm{CritPts} = \mathrm{msolve}(I)$
2. $\{p_1, p_2\} = C \cap \mathrm{CritPts}$
3. **return** $\{c_1, c_2\} = \mathrm{pr}_{x_1}(\{p_1, p_2\})$.



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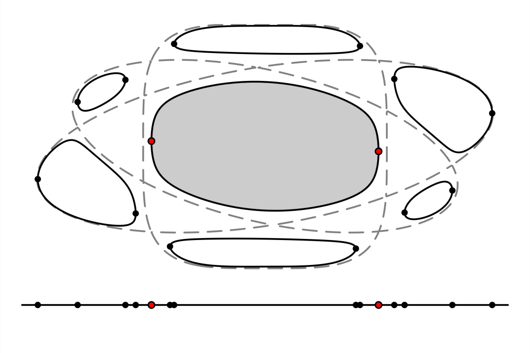
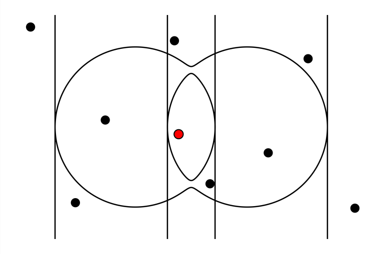


Case $\dim I > 0$:

1. Compute $\langle h(x_1) \rangle = I \cap \mathbb{Q}[x_1]$.
2. $\mathrm{Pts} = \mathrm{HypersurfaceRegions.jl}(F_t \cdot h(x_1))$.
3. $\mathrm{InPts} = \mathbf{pr}_{x_1}(C_t \cap \mathrm{Pts})$
4. **return** $\{\ c_1 = \mathbf{max}\{c \in h^{-1}(0) \mid c < \mathrm{InPts}\},$ $\quad c_2 = \mathbf{min}\{c \in h^{-1}(0) \mid c > \mathrm{InPts}\}\ \}$



Integration Ideals

Let $D_{t,\mathbf{x}} = \mathbb{Q}[t,\mathbf{x}]\langle \partial_t, \partial_{x_1}, \ldots, \partial_{x_n} \rangle$ the Weyl algebra of linear differential operators with polynomial coefficients in t and $x_1, \ldots, x_n$.

Fact: Let $J \subset D_{t,\mathbf{x}}$ be an annihilating ideal of a rational function $A \in \mathbb{Q}(t,\mathbf{x})$. Then, the *integration ideal*

$$\mathcal{I}_t(J) := (J + \partial_{x_1} D_{t,\mathbf{x}} + \cdots + \partial_{x_n} D_{t,\mathbf{x}}) \quad \cap \quad D_t \tag{1}$$

annihilates periods of A that depend on the parameter t.

Theorem
Integration ideals of holonomic ideals are holonomic (and hence non-zero).

Note: Elimination over $R_n := \mathbb{Q}(t,x)\langle \partial_t, \partial_1, \ldots, \partial_n \rangle$ can lead to wrong results [4].

[4] Mezzarobba and Safey El Din. 2023. *Volumes.* A Git repository.

Creative Telescoping Algorithms

Previous work (non-exhaustive)

The method of creative telescoping was first presented by Zeilberger (1990)

Gröbner basis approaches (holonomy)

- Integration of holonomic functions (Takayama 1990, Oaku-Takayama 1997, Chyzak-Salvy 1998)
- Integration of holonomic functions over semi-algebraic sets (Oaku 2013)

Algorithms based on Ansätze and resolutions of linear differential equations (D-finite)

- Univariate integration of D-finite functions (Chyzak 2000)
- Fast heuristic for univariate integration of D-finite functions (Koutschan 2010)

Reduction-based approaches (D-finite)

- Univariate integration of bivariate rational functions (Bostan-Chen-Chyzak-Li 2010)
- Multivariate integration of rational functions (Bostan-Lairez-Salvy 2013, Lairez 2016)
- Univariate integration of D-finite functions (van der Hoeven 2018, Bostan-Chyzak-Lairez-Salvy 2018, Chen-Du-Kauers 2023)

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[Brochet, June 10th, 2026: *Multivariate Creative Telescoping and Applications to k-regular graphs counting.* MPI MiS.]

Recent approach by Brochet–Chyzak–Lairez [5] working over $\mathbb{Q}(t)[x]$.

[5] Hadrien Brochet, Frédéric Chyzak, and Pierre Lairez. 2025. *Faster multivariate integration in D-modules.* arXiv:2504.12724

Computation Time (non-parallelized implementation)

SageMath Implementation (+ MCT.jl, + HypersurfaceRegions.jl):

`https://github.com/nicolas-alexander-weiss/exact_convex_volumes`

Convex Body	Dimension	Degree	Prec	**Time**	gCAD and Symbolic	+ Numeric
Triangle	2	3	50	0.9s		
Tetrahedron	3	4	50	6.8s		
Ellipse and circle	2	4	50	2m 15.0s		
Two ℓ_4-balls	2	8	50	10m 10s		
Two ℓ_4-balls	4	8	95	15h		
			100			

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Tetrahedron	3	4	50	6.8s	1s	
Ellipse and circle	2	4	50	2m 15.0s	3.5s	
Two ℓ_4-balls	2	8	50	10m 10s		0.5m
Two ℓ_4-balls	4	8	95	15h		2.5m
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Two ℓ_4-balls	2	8	50	10m 10s		0.5m
Two ℓ_4-balls	4	8	95	15h		2.5m
$\{f(x,y) > 0\}$	2	6	50	0.8s	x	10.4s
$\{f(x,y) > 0\}$	2	6	100	0.9s	x	12.3s
$\{f(x,y) > 0\}$	2	6	200	1s	x	16.9s

Where $f(x,y) = 1 - (x^2 + y^2)^3 - 3/10(x^4y^2 + x^2y^4)$.

Thank you for listening! Questions?

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