Exact Volumes of Semi-Algebraic Convex Bodies

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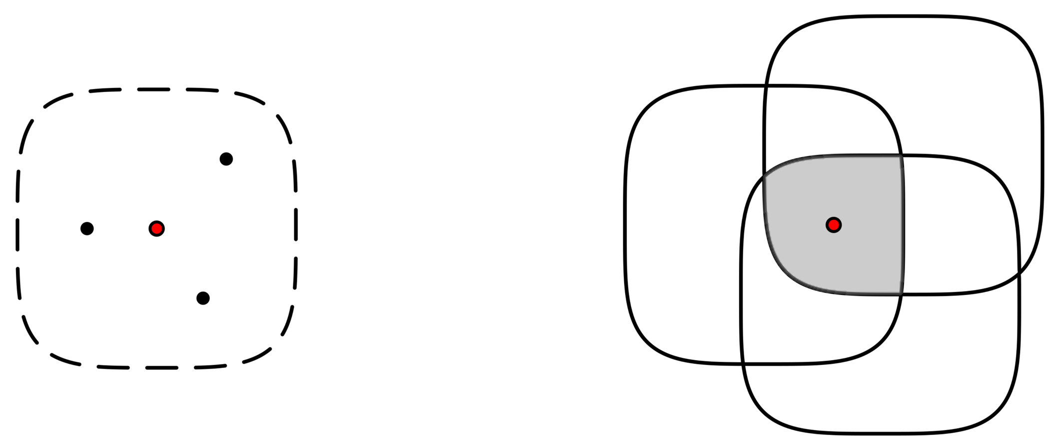
A Problem from Geometric Statistics

Let $K = \{\mathbf{x} \in \mathbb{R}^n \mid 1 - x_1^p - \ldots - x_n^p > 0\}$, $p \in 2\mathbb{N}$. Given samples $X_1, \ldots, X_k$ from a uniform distribution on $K + \theta$ with $\theta \in \mathbb{R}^n$, determine θ:



$\Rightarrow$ The **M**aximum **L**ikelihood **E**stimator set $= \bigcap_i K + X_i$, as in [1], is guaranteed to contain θ.
Problem: Quantify **uncertainty**$(\theta \mid X_1, \ldots, X_k)$ by determining **Vol** $(\bigcap_i K + X_i)$ with arbitrary precision.

Slice Volumes are Periods as functions in t

Let $f \in \mathbb{Q}[t, \mathbf{x}]$ and $V(f)$ smooth. Let C be union of fiberwise bdd. conn. components of $\{(t, \mathbf{x}) \in \mathbb{R}^{1+n} \mid f(t, \mathbf{x}) > 0\}$. Assume the slice $C_t = C \cap \{t\} \times \mathbb{R}^n$ to be smooth.

$$\mathrm{Vol}(C_t) = \int_C 1 d\mathbf{x} = \underbrace{\int_{\partial C} x_1 dx_2 \ldots dx_n}_{\text{Stokes' Thm.}} = \underbrace{\frac{1}{2\pi i} \int_{\Gamma_t} \frac{(\partial_{x_1} \bullet f) x_1}{f} d\mathbf{x}}_{\text{Leray's Residue Thm.}}$$

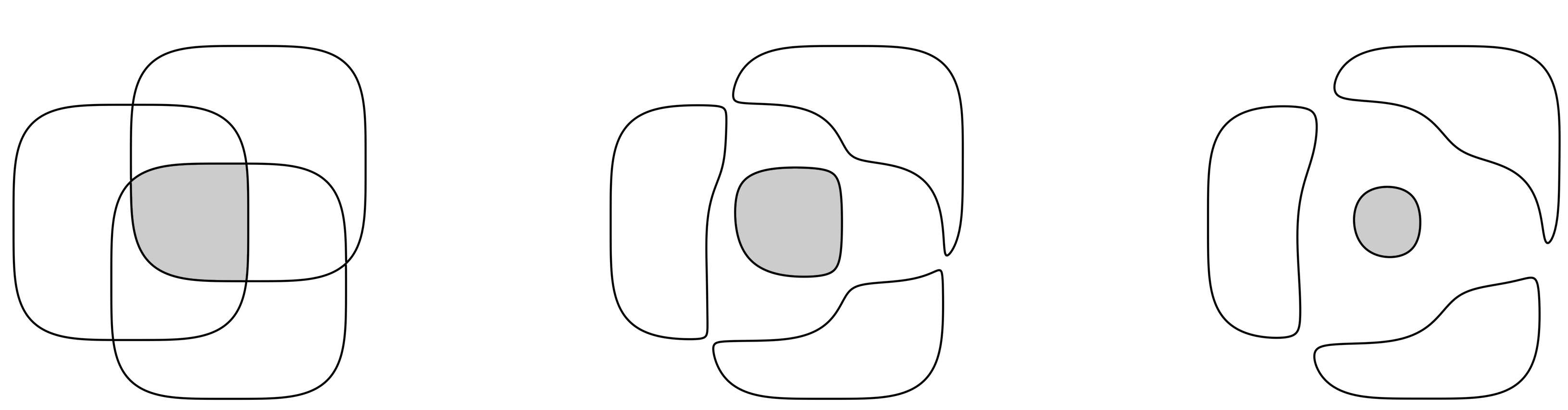
Proposition LMS'19, [2]: $\mathrm{Vol}(C_t)$ *is a solution of a* ***Picard–Fuchs*** *operator* $P_t \in \mathbb{Q}[t]\langle \partial_t \rangle$ *for t in any interval of adjacent critical values of the projection*

$$\mathrm{pr}_t : V(f) \to \mathbb{R}, \qquad (t, \mathbf{x}) \mapsto t.$$

Slogan [2]: *Volumes can be computed by solving a linear ODE to* ***arbitrary precision***. [2]

Volumes of Semi-Algebraic Sets: Deformation

Deform a compact n-dimensional semi-algebraic set $C = \bigcap_i \{\mathbf{x} \in \mathbb{R}^n \mid f_i(\mathbf{x}) > 0\}$ into smooth $C_t = C \cap \{\mathbf{x} \in \mathbb{R}^n \mid \prod_i f_i(\mathbf{x}) - t > 0\}$:



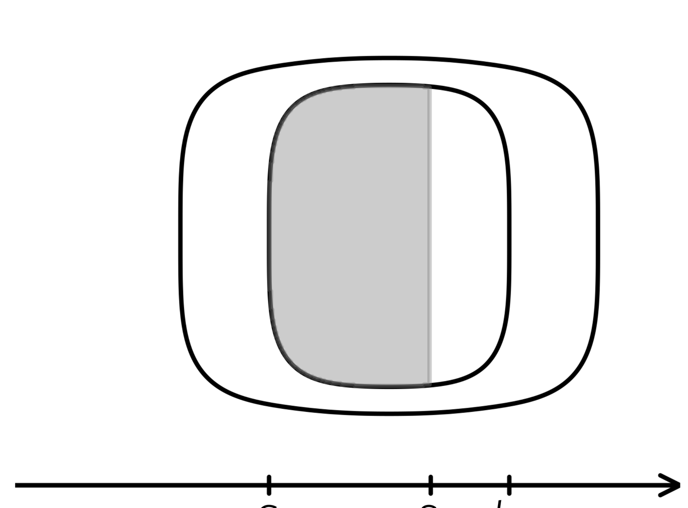
(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.5$

Compute P_t for $F_t := \prod_i f_i(\mathbf{x}) - t$. Solve on $0 < t << 1$ and take limit $t \to 0$.

Initial Data: *Volumes* $Vol(C_{t_0})$ *with smooth boundary* $F(t = t_0, \mathbf{x})$.

Volumes of Semi-Algebraic Sets: Integration

Let $C \subset \{x \in \mathbb{R}^n \mid f(\mathbf{x}) > 0\}$ compact. $V(f)$ smooth. Set $t = x_1$. Obtain the volume by integrating $\phi(t) = \mathrm{Vol}(C \cap \{t\} \times \mathbb{R}^{n-1})$ on the intervals of adjacent critical values.



One can solve the integration problem

$$\mathrm{Vol}(C) = \int_a^b \mathrm{Vol}(C \cap \{t = s\}) ds$$

by solving the linear ODE $P_t \partial_t \bullet \phi(t) = 0$ and evaluating at $t = b$.

Initial Data / Recursion: $Vol(C_{t_0})$ *are smooth volumes of dimension* $n - 1$.

Deformation of Semi-Algebraic Convex Bodies

The ℓ_p-polynomials are *concave functions*. This setting allows to simplify the approach by [2].

Proposition (Ramesh–W '26, [?])
Let $f_1, \ldots, f_k \in \mathbb{Q}[x_1, \ldots, x_n]$ *be concave functions. Then, its deformation*

$$C_t := \{x \in \mathbb{R}^n \mid \prod_i f_i(x) - t > 0\} \cap C$$

is a convex set for all $t > 0$.

As a **Corollary**, there is a unique interval of critical values coming from ∂C_t on which initial data have to be computed at each step of the recursion.

Identifying the Critical Values of $\partial C_t \subset V(F_t)$

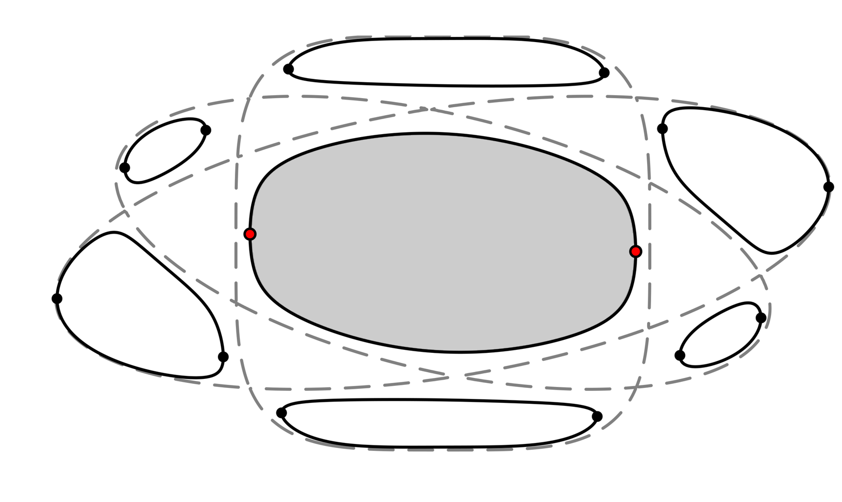
Project onto x_1. $F_t := \prod_i f_i - t$. Critical locus $I := \langle F_t, \partial_2 F_t, \ldots, \partial_n F_t \rangle \subset \mathbb{Q}[\mathbf{x}]$.

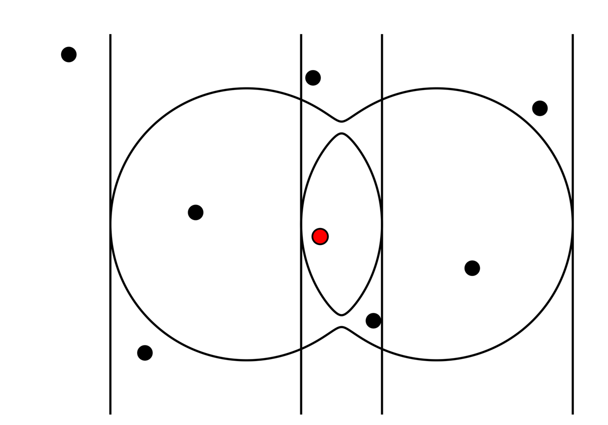
Case $\dim I = 0$:

1. CritPts = msolve(I)
2. $\{p_1, p_2\} = C \cap$ CritPts
3. **return** $\{c_1, c_2\} = \mathrm{pr}_{x_1}(\{p_1, p_2\})$

Case $\dim I > 0$:

1. Compute $\langle h(x_1) \rangle = I \cap \mathbb{Q}[x_1]$.
2. Pts = HypersurfaceRegions.jl$(F_t \cdot h(x_1))$.
3. InPts $= \mathrm{pr}_{x_1}(C_t \cap \text{Pts})$
4. **return** $\{c_1 = \max\{c \in h^{-1}(0) \mid c < \text{InPts}\},$ $c_2 = \min\{c \in h^{-1}(0) \mid c > \text{InPts}\}\ \}$





Example

For illustration of the approach, consider the intersection C of two ℓ_4 balls in $\mathbb{R}^2$ with centers $(0, 0)$ and $(0, 1/2)$ defined by

$$f_1(x, y) = 1 - x^4 - y^4 \quad \text{and} \quad f_2(x, y) = 1 - (x - 1/2)^4 - y^4.$$

The volume of the deformation $C_t = C \cap \{f_1 \cdot f_2 - t > 0\}$ is annihilated by an operator of order $\mathrm{ord}(P_t) = 6$ and degree $\deg(P_t) = 18$.

For the intial data, we hence compute six further Picard–Fuchs operators P_x each of order $\mathrm{ord}(P_x) = 2$ and degree $\deg(P_x) = 23$.

Finally, using a total of 12 one-dimensional slice volumes, we determine the volume of C to be $\mathrm{Vol}(C) = 2.7083448262997200900018449369806437205\ldots$

Try the implementation [3] (Zenodo / Github).

References

[1] V. Koltchinskii, L. Ramesh, and M. Wahl. *Maximum likelihood estimation of the location of a symmetric convex body*. In prepration, 2026.

[2] P. Lairez, M. Mezzarobba, and M. Safey El Din. *Computing the Volume of Compact Semi-Algebraic Sets*. In *Proceedings of ISSAC '19*. ACM, 2019.

[3] L. Ramesh and N. Weiss. *nicolas-alexander-weiss/exact_convex_volumes: v0.0.1*. https://doi.org/10.5281/zenodo.18472470

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