Weight-Shifting and Kinematic Flow

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Setup

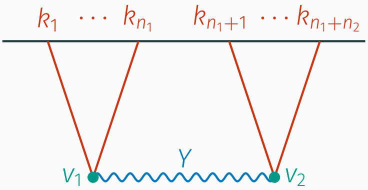
Setup

- We will be considering rigid D-dimensional **de Sitter space** with two fields with a simple polynomial interaction

$$S = \int \frac{\mathrm{d}^D x}{2H^{D-2}\eta^D}\left[\eta^2\partial_\eta\varphi\partial_\eta\varphi - \eta^2\partial_i\varphi\partial^i\varphi - \frac{D(D-2)}{4}\varphi^2 \right.$$
$$\left. + \eta^2\partial_\eta\chi_\nu\partial_\eta\chi_\nu - \eta^2\partial_i\chi_\nu\partial^i\chi_\nu + \left(\nu^2 - \frac{(D-1)^2}{4}\right)\chi_\nu^2 + \frac{2\lambda}{n!}\varphi^n\chi_\nu\right]$$

- We want to compute **correlation functions** of these fields evaluated on the **future boundary** at the end of inflation (these correlations source all the structure in the universe)
- As an intermediate step we will consider the so called **coefficients of the *wavefunction of the universe*** from which we can straightforwardly find these correlators

Feynman Rules



Feynman Integral

(up to scaling by constants)

Bulk-to-bulk propagator

$$I = \int \frac{\mathrm{d}\eta_1}{(\eta_1)^D} \frac{\mathrm{d}\eta_2}{(\eta_2)^D} \left(\prod_{a=1}^{n_1} K_{\frac{1}{2}}(k_a, \eta_1) \right) G_\nu(Y, \eta_1, \eta_2) \left(\prod_{b=n_1+1}^{n_1+n_2} K_{\frac{1}{2}}(k_b, \eta_2) \right)$$

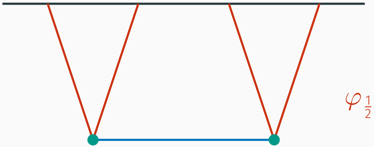
Bulk-to-boundary propagator

- $\eta \in \mathbb{R}_{<0}$ conformal time, $\vec{k}_e \in \mathbb{R}^{D-1}$ momentum, $k_e := |\vec{k}_e| \in \mathbb{R}_{\geq 0}$

Goal: Closed analytic expression, or at least “good” / “structured” diff. equation.

Conformally Coupled vs Massive

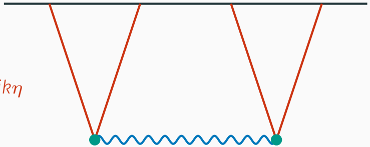
Conformally coupled



$$\chi_{\frac{1}{2}}(k,\eta) = -\frac{iH(-\eta)^{\frac{D-2}{2}}}{\sqrt{2k}} e^{ik\eta}$$

$$\varphi_{\frac{1}{2}}(k,\eta) = -\frac{iH(-\eta)^{\frac{D-2}{2}}}{\sqrt{2k}} e^{ik\eta}$$

Massive



$$\chi_{\nu}(k,\eta) = \frac{\sqrt{\pi}H}{2} e^{i\theta}(-\eta)^{\frac{D-1}{2}} H_{\nu}^{(2)}(-k\eta)$$

The propagators

$$K_{\frac{1}{2}}(k,\eta) = \frac{(-\eta)^{\frac{D-2}{2}}}{(-\eta_0)^{\frac{D-2}{2}}} e^{ik(\eta-\eta_0)}, \qquad G_{\nu}^{>}(Y,\eta,\eta') = \chi_{\nu}^{*}(Y,\eta)\chi_{\nu}(Y,\eta')\theta(\eta-\eta')$$

$$G_{\nu}(Y,\eta,\eta') = G_{\nu}^{>}(Y,\eta,\eta') + G_{\nu}^{>}(Y,\eta',\eta) - \chi_{\nu}(Y,\eta)\chi_{\nu}(Y,\eta')\frac{\chi_{\nu}^{*}(Y,\eta_0)}{\chi_{\nu}(Y,\eta_0)}$$

A Pfaffian System

An Unmotivated Basis

- Rather than working directly with the massive modefunctions we introduce new functions

$$h_\nu^\pm(z) = \frac{\sqrt{\pi}H}{2}\left[z^\nu H_\nu^{(2)}(z) \pm i\partial_z\left(z^\nu H_\nu^{(2)}(z)\right)\right]$$

- We then build propagators from this (on the complementary series, $\nu \in \mathbb{R}_{\geq 0}$)

$$G_>^{\pm\pm}(Y,\eta,\eta') = h_\nu^\mp(-Y\eta)^* h_\nu^\pm(-Y\eta')\theta(\eta-\eta')$$
$$G_\nu^{\pm\pm}(Y,\eta,\eta') = G_>^{\pm\pm}(Y,\eta,\eta') + G_>^{\pm\pm}(Y,\eta',\eta) + h_\nu^\pm(-Y\eta)h_\nu^\pm(-Y\eta')$$
$$G_\nu^0 = i(-Y\eta)^{2\nu}\,\delta(\eta-\eta')$$

- We then define from this an integral which forms the basis of our differential equations

$$I_\nu^a = \int \mathrm{d}\eta_1\mathrm{d}\eta_2(-\eta_1)^{-\alpha_1-\delta-1}(-\eta_2)^{-\alpha_2-\delta-1}e^{iX_1\eta_1}e^{iX_2\eta_2}G_\nu^a(Y,\eta_1,\eta_2), \quad a = \pm\pm, 0$$

$$I_\nu = I_\nu^{++} + I_\nu^{+-} + I_\nu^{-+} + I_\nu^{--}, \qquad \alpha_i = \frac{D}{2} + \frac{1}{2}n_i(2-D), \qquad \delta = \nu - \frac{1}{2}$$

A rank 5 system

Massive internal edge (i.e. $\nu \in \mathbb{R}_{\geq 0}$ generic) [Baumann et. al., *in preparation.*]

$$
\begin{aligned}
\mathrm{d}I_\nu^{++} &= \left(\alpha_1\ell_1^+ + \alpha_2\ell_2^+\right) I_\nu^{++} + \delta\ell_2^+\, I_\nu^{+-} + \delta\ell_1^+\, I_\nu^{-+} \\
\mathrm{d}I_\nu^{+-} &= \left(\alpha_1\ell_1^+ + \alpha_2\ell_2^-\right) I_\nu^{+-} + \delta\ell_2^-\, I_\nu^{++} + \delta\ell_1^+\, I_\nu^{--} + 2(\ell_1^+ - \ell_2^-) I_\nu^0 \\
\mathrm{d}I_\nu^{-+} &= \left(\alpha_1\ell_1^- + \alpha_2\ell_2^+\right) I_\nu^{-+} + \delta\ell_1^-\, I_\nu^{++} + \delta\ell_2^+\, I_\nu^{--} + 2(\ell_2^+ - \ell_1^-) I_\nu^0 \\
\mathrm{d}I_\nu^{--} &= \left(\alpha_1\ell_1^- + \alpha_2\ell_2^-\right) I_\nu^{--} + \delta\ell_1^-\, I_\nu^{+-} + \delta\ell_2^-\, I_\nu^{-+} \\
\mathrm{d}I_\nu^{0} &= (\alpha_1 + \alpha_2)\ell\, I_\nu^0
\end{aligned}
$$

Where the letters are

$$
\ell_1^\pm = \mathrm{d}\log\left(\frac{X_1}{Y} \pm 1\right), \qquad \ell_2^\pm = \mathrm{d}\log\left(\frac{X_1}{Y} \pm 1\right), \qquad \ell = \mathrm{d}\log\left(\frac{X_1 + X_2}{Y}\right).
$$

Kinematic Flow: There exist diagrammatic rules which generalise this to arbitrary correlators

A rank 3+1 system

Conformally coupled internal edge (i.e. $\nu = 1/2$)

$$
\begin{aligned}
\mathrm{d}I^{++}_{\frac{1}{2}} &= \left(\alpha_1\ell_1^+ + \alpha_2\ell_2^+\right) I^{++}_{\frac{1}{2}} + \delta\ell_2^+\, I^{+-}_{\frac{1}{2}} + \delta\ell_1^+\, I^{-+}_{\frac{1}{2}} \\
\mathrm{d}I^{+-}_{\frac{1}{2}} &= \left(\alpha_1\ell_1^+ + \alpha_2\ell_2^-\right) I^{+-}_{\frac{1}{2}} + \delta\ell_2^-\, I^{++}_{\frac{1}{2}} + \delta\ell_1^+\, I^{--}_{\frac{1}{2}} + 2(\ell_1^+ - \ell_2^-)\, I^{0}_{\frac{1}{2}} \\
\mathrm{d}I^{-+}_{\frac{1}{2}} &= \left(\alpha_1\ell_1^- + \alpha_2\ell_2^+\right) I^{-+}_{\frac{1}{2}} + \delta\ell_1^-\, I^{++}_{\frac{1}{2}} + \delta\ell_2^+\, I^{--}_{\frac{1}{2}} + 2(\ell_2^+ - \ell_1^-)\, I^{0}_{\frac{1}{2}} \\
\mathrm{d}I^{--}_{\frac{1}{2}} &= \left(\alpha_1\ell_1^- + \alpha_2\ell_2^-\right) I^{--}_{\frac{1}{2}} + \delta\ell_1^-\, I^{+-}_{\frac{1}{2}} + \delta\ell_2^-\, I^{-+}_{\frac{1}{2}} \\
\mathrm{d}I^{0}_{\frac{1}{2}} &= (\alpha_1 + \alpha_2)\,\ell\, I^{0}_{\frac{1}{2}}
\end{aligned}
$$

Where the letters are

$$
\ell_1^\pm = \mathrm{d}\log\left(\frac{X_1}{Y} \pm 1\right), \qquad \ell_2^\pm = \mathrm{d}\log\left(\frac{X_1}{Y} \pm 1\right), \qquad \ell = \mathrm{d}\log\left(\frac{X_1 + X_2}{Y}\right).
$$

This system is smaller, sparser and can be solved with a sourced first order differential equation

Hyperplane Arrangement

Integral Representation

- We now introduce integral representations to make all time dependence exponential

$$(-\eta)^{-\gamma-1} = a_\gamma \int_0^\infty \mathrm{d}\omega \omega^\gamma e^{i\omega\eta}, \quad h_\nu^\pm(z) = -ib_\nu \int_\infty^1 \mathrm{d}t \frac{(t^2-1)^\delta}{1\mp t} e^{-itz}$$

- This puts all time integrals into the trivial form

$$\int_{-\infty}^{\eta'} \mathrm{d}\eta e^{iB\eta} = \frac{1}{iB} e^{iB\eta'}, \quad B_i = \frac{X_i}{Y} + \omega_i + t_i, \quad B_\pm = \frac{X_1}{Y} + \frac{X_2}{Y} + \omega_1 + \omega_2 \pm (t_1 - t_2)$$

- Our basis elements thus become

$$I_\nu^{\pm\pm} \propto Y^{\alpha_1+\alpha_2+2\delta} \int \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}t_1 \mathrm{d}t_2 \, \omega_1^{\alpha_1+\delta} \omega_2^{\alpha_2+\delta} (t_1^2-1)^\delta (t_2^2-1)^\delta$$
$$\times \left(\frac{1}{(1\pm t_1)(1\mp t_2) B_2 B_-} + \frac{1}{(1\mp t_1)(1\pm t_2) B_1 B_+} - \frac{1}{(1\mp t_1)(1\mp t_2) B_1 B_2} \right)$$

Integral Representation II

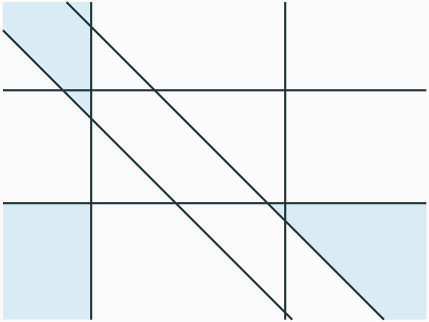


Figure 1: The 4-dimensional hyperplane arrangement in the massive (ν generic) case. Above the slice is shown for $t_1 = 2.5$ and $t_2 = 1.2$.

From the general GKZ setup:

$$\text{rank}_{GKZ} = \chi(\mathcal{A})$$

Counting bounded chambers: [Huh, 2013]

$$\chi(\mathcal{A}) = \#\left\{0 = d_{w,t}\log(w_i^{r_i}(t_i \pm 1)^{s_i} B_1^{\nu_1} B_2^{\nu_2} B_+^{\nu_3} B_-^{\nu_4}\right\}$$
$$= \quad 44$$

In comparison, for our basis: rank $= 5$!

Weight-Shifting

Weight-Shifting: Abstractly

Given a function $h_\nu(x)$, one may study its annihilator in the algebra of *difference-differential operators*

$$DD := \mathbb{C}[\nu]\langle \sigma_\nu, \sigma_\nu^{-1} \rangle \otimes_{\mathbb{C}} \mathbb{C}[x]\langle \partial_x \rangle$$

Shift Algebra S_ν — Weyl Algebra D_x

Let $\mathrm{Ann}_{DD}(h_\nu(x))$ and $\mathrm{Ann}_{D_x(\nu)}(h_\nu(x))$ denote the respective annihilators.

Definition
A weight-shifting operator for $h_\nu(x)$ is a differential operator $P \in D_x(\nu)$, such that

$$\sigma_\nu - P \in \mathrm{Ann}_{DD}(h_\nu(x))$$

[i.e. $P \bullet h_\nu(x) = \sigma_\nu \bullet h_\nu(x) = h_{\nu+1}(x)$]

Remark: The set of weight shifting operators is affine over $D_x(\nu)$.

Weight-Shifting: Motivation

- Particles with weights that differ from conformally coupled by an integer are special:
  - They are physically interesting
  - In momentum space their propagators are very simple
  - They show interesting structure (conservation/enhanced soft limits)
- In this differential equations language this simplicity comes from the fact that they are related to the conformally coupled case by a shift operator
- This reduces the order of the differential equation that we need to solve

Question: Can we find this **weight-shifting operator** to go from $\nu = 1/2$ to $\nu = 3/2$ and uplift our knowledge of this simpler problem?

Weight-Shifting: Implementation

- The weight-shifting operator for the full wavefunction is known and is a second order differential operator [Arkani-Hamed et. al, 2018]
- Equipped with our differential system we can convert this to a matrix:

$$R_{\frac{1}{2}} \cdot I_{\frac{1}{2}} = A \begin{pmatrix} L_1^+ L_2^+ & L_1^+ \ell_2^- & \ell_1^- L_2^+ & \ell_1^- \ell_2^- & -4\ell \\ L_1^+ \ell_2^+ & L_1^+ L_2^- & \ell_1^- \ell_2^+ & \ell_1^- L_2^- & -4\ell \\ \ell_1^+ L_2^+ & \ell_1^+ \ell_2^- & L_1^- L_2^+ & L_1^- \ell_2^- & -4\ell \\ \ell_1^+ \ell_2^+ & \ell_1^+ L_2^- & L_1^- \ell_2^+ & L_1^- L_2^- & -4\ell \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{\frac{1}{2}}^{++} \\ I_{\frac{1}{2}}^{+-} \\ I_{\frac{1}{2}}^{-+} \\ 0 \\ I_{\frac{1}{2}}^{0} \end{pmatrix} = I_{\frac{3}{2}}$$

Where we have slightly abused our previous notation to write

$$\ell_i^{\pm} = \frac{X_i}{Y} \pm 1, \qquad \ell = \frac{X_1 + X_2}{Y}, \qquad L_i^{\pm} = \frac{X_i}{Y} \pm (2\alpha_i - 1), \quad A = -\frac{1}{4(1-\alpha_1)(1-\alpha_2)}$$

Next Steps

- There is still much to understand about this plane arrangement and the geometry
- We would like to extend the weight-shifting to arbitrary massless edges
- A similar differential structure exists for shifting to other interactions and spinning fields, it should be possible to understand this as a basis rotation too
- We could shift repeatedly to arbitrarily high values of ν, this could be an interesting way to explore the large mass limit of this system

Thank you for listening!
Questions?